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Computational Design of Turbomachine Blades

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Based upon Wu's general three-dimensional turbomachine flow theory, a set of computer programs was established to measure flow along the relative stream surfaces S_1 and S_2, employing nonorthogonal curvilinear coordinates and nonorthogonal velocity components. The streamline extension technique, matrix direct solution, and line relaxation method are used in these programs. With the employment of fictitious grid points, the internal storage capacity required in the matrix direct solution is reduced by nearly two-thirds. The matrix direct solution and the line relaxation method are found to converge nearly 10 times faster than the streamline extension technique. These programs constitute a design system of axial flow turbomachine blades based upon three-dimensional flow. Experience shows that iterative solutions considering three-dimensional flow obtained by the successive calculation of the S_1 and S_2 surface flows are rapid and practical. The major results of a number of typical applications are included in the paper.

Nomenclature

a_{ij} = basic metric tensor
I = stagnation rothalpy
$\dot{q}$ = heat transfer to gas per unit mass per unit time
s = entropy of gas per unit mass
s_i = length of coordinate line, x^i
W = relative velocity of gas
w^i = physical component of W tangent to x^i
x^i = nonorthogonal curvilinear coordinates
α = relaxation factor
ζ = $(\rho W_l \tau_2)_m / (\rho W_l \tau_2)_i$ (see Fig. 8)
Θ_{ij} = angle included by the coordinate lines
μ = tilt angle (see Fig. 4)
Π' = viscous force tensor
τ_1 = normal thickness of S_1 stream filament
τ_2 = circumference thickness of S_2 stream filament
Φ = dissipation function
Ψ = stream function
ω = angular velocity of blade
$\partial/\partial x^i$ = partial differential along stream surface

Superscript

0 = stagnation state

Subscripts

i = at inlet
∞ = far upstream of blade row

Introduction

SINCE its publication in the early 1950's, Wu's general theory of three-dimensional turbomachine flow has been widely used to various degrees in the design and analysis of turbomachines.[1] The development of the employment of nonorthogonal curvilinear coordinates and the corresponding nonorthogonal velocity components in the 1960's[2-5] enhances the convenience, accuracy, and speed of the numerical solution. Computer programs for the S_1 and S_2 relative flow surfaces based upon a new set of equations by the methods of streamline extension, line relaxation, and matrix direct solution are described and compared in this paper. Some results of typical applications are also included.

Basic Aerothermodynamic Relations

The continuity equation in terms of the relative velocity is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho W) = 0 \tag{1}$$

Newton's second law and the first and second law of thermodynamics for a viscous fluid are[5]

$$-\frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla \Pi' = \frac{\mathrm{d}'W}{\mathrm{d}t} - \omega^2 r + 2\omega \times W \tag{2}$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{1}{\rho}\frac{\partial p}{\partial t} + \dot{q} + \frac{1}{\rho}\nabla \cdot (\Pi' W) \tag{3}$$

$$T\frac{\mathrm{d}s}{\mathrm{d}t} = \dot{q} + \frac{\Phi}{\rho} \tag{4}$$

For engineering design calculation the gas motion is usually considered to be steady relative to the blade. Then, Eq. (1) becomes

$$\nabla \cdot (\rho W) = 0 \tag{5}$$

Since the work done by a fluid element against the viscous forces acting upon it by the surrounding fluid is approximately equal to the heat transfer to the fluid element, it is usually taken to be

$$\frac{\mathrm{d}I}{\mathrm{d}t} = 0 \tag{6}$$

which means that the stagnation rothalpy I remains constant along a relative streamline. For the dynamic equation (2), the viscous force in the main flow is small, and its effect on the velocity distribution near the end wall can be partly taken into account by the entropy term in the following form of Eq. (2) (see Fig. 1):

$$-W \times [\nabla \times V] = -\nabla I + T\nabla s \tag{7}$$

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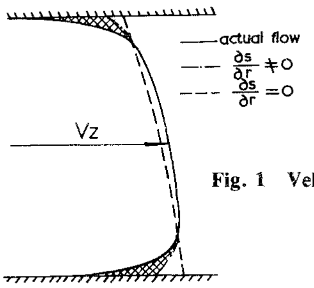

Fig. 1 Velocity distribution in radial direction.

The entropy increase across a blade row can be conveniently calculated by the use of some empirical data.

From Fig. 1 it can be seen that the calculated mass flow should be slightly greater than the actual mass flow; therefore, an appropriate flow coefficient K_G is considered to be as follows:

$$G_{\text{cal}} = K_G G_{\text{actual}} \tag{8}$$

Use of Nonorthogonal Curvilinear Coordinates and Nonorthogonal Velocity Components

The choice of a coordinate system directly affects the form of the equations to be solved, the expression of the boundary conditions, and the programming. It has undergone a process of development. For general three-dimensional studies, the geometry of the turbomachine naturally led to the use of the cylindrical coordinates (r, φ, z) such as reported in Ref. 1. Then, the meridional coordinate l together with the relative angular coordinate φ was found convenient, especially for calculations along a general S_1 stream surface of revolution. Later, requirements for a standard universal program usable for arbitrary blade shape led to the use of nonorthogonal grid system (η,ξ) as first suggested in Ref. 2. The orthogonal velocity component system was still retained; however, usually it could not perfectly satisfy the tangency condition at the boundary. Therefore, basic studies on the use of the nonorthogonal curvilinear coordinate system together with the use of the corresponding nonorthogonal velocity components was begun by Prof. Wu in 1963,[3] in which the tensor method was employed to obtain the most general set of governing equations for S_1 and S_2 surfaces. These equations are not much more complicated than those given in Ref. 1, which employ the orthogonal curvilinear coordinate system, and possess some obvious advantages. For example, in design calculations for the central S_2 surface, an entirely arbitrary set of nonorthogonal curvilinear coordinates (x^1, x^2) can be chosen in the meridional plane, such as shown in Fig. 2. To analyze the flow past an arbitrary cascade of blades on a given S_1 surface of revolution, it is convenient to choose this given surface as a x^3 = constant surface and choose an arbitrary (x^1, x^2) set on this surface. In general, this kind of coordinate system can fit the arbitrary shape of the boundaries very well, the expression of the boundary condition is simple and clear, the grid pattern can be standardized (Fig. 3), and the grid spacing can be chosen at will to increase the accuracy of the computation. In addition, since the velocity components are tangent to the coordinate lines, at the boundary there exists only the velocity component along the direction of the boundary coordinate line, the component corresponding to the other coordinate line being equal to zero. The tangency condition at the boundary is thus satisfied exactly.

The principal equation to be solved is more complicated than the one in the orthogonal system, but not much more so. Also, once obtained, the equation is general, no matter how the coordinates (x^1, x^2) are selected.

With the use of a modern computer, the solution of a somewhat more complicated equation does not lead to any difficulty, whereas better simulation of conditions at the boundaries does improve the convergence and accuracy of the solution. Therefore, starting in 1969, our computer programs for general design and analysis calculation of turbomachine flow worked entirely with this new system of nonorthogonal curvilinear coordinates, together with the corresponding nonorthogonal velocity components.

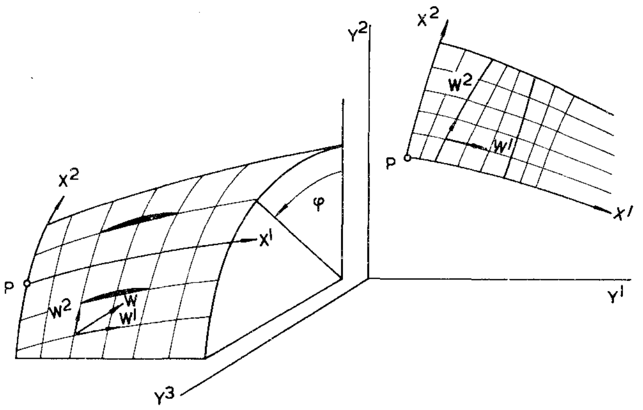

Fig. 2 Nonorthogonal curvilinear coordinates and nonorthogonal velocity components used for S_1 and S_2 stream surfaces.

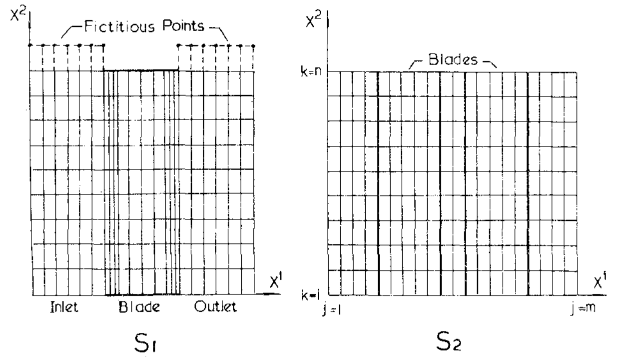

Fig. 3 Standardized grid pattern and fictitious points.

We found in the programming work that it was convenient to put into the computer the boundary coordinates in the usual cylindrical coordinate system (r, φ, z) and to compute the components of the basic metric tensor and angle included in the coordinate lines by numerical differentiation.

These programs are probably the first which the present system has ever used in engineering calculations. We found that they worked very well and that we may find many more applications in other fields of engineering.

Computer Program for Relative Stream Surface S_2

Calculations of the flow along the two kinds of relative stream surfaces S_1 and S_2 form the basis of the three-dimensional turbomachine flow calculations. As the calculations take into account the appropriate short distance τ_1 or τ_2 between two neighboring stream surfaces, the calculations are actually those of thin stream filaments of which the calculated surfaces correspond to the central surface of the filament. Two methods of solution are generally employed, as described in the following sections.

Streamline Extension Method (or Streamline Curvature Method)

It was first suggested in Ref. 7 that the radial equilibrium equation may be used to calculate axial velocity distribution in the radial direction. A term representing the curvature effect of the streamline projection in the meridional plane is involved in this calculation. In Ref. 8, the mean streamline method was suggested to compute the circumferential variation of the velocity components. Terms representing the effect of the streamline curvature and the thickness of blade are involved in the calculation.

In the case of the S_2 surface, with the use of the present coordinate and velocity component system, the principal equation is now (for derivation, see Ref. 3)

$$\frac{\partial(W^1\sqrt{a_{11}})}{\partial x^2}=\frac{\sqrt{a_{11}}}{W^1}\left[-\frac{W_\varphi}{r}\frac{\partial(V_\theta r)}{\partial x^2}+\frac{\partial I}{\partial x^2}-T\frac{\partial s}{\partial x^2}-f_2\right]$$
$$+\frac{\partial}{\partial x^1}\left[(W^1\cos\theta_{12}+W^2)\sqrt{a_{22}}\right]-\frac{\partial}{\partial x^2}\left[W^2\sqrt{a_{11}}\cos\theta_{12}\right] \tag{9}$$

It is seen that this equation is quite similar to the radial equilibrium equation of Ref. 7, but somewhat more complicated. In the program, all the terms on the right side of the equation are to be corrected by the result obtained in the preceding cycle. As in Ref. 7, a value of W^1 is assumed at the hub to start the calculation. The hub value of W^1 is corrected until mass flow through the entire passage is satisfied. In order to avoid this inconvenience, the continuity equation is first used to provide the following relations:

$$\frac{\partial\Psi}{\partial x^2}=\tau\rho W^1\sqrt{a_{22}}\sin\Theta_{12} \tag{10a}$$

$$\frac{\partial\Psi}{\partial x^1}=-\tau\rho W^2\sqrt{a_{11}}\sin\Theta_{12} \tag{10b}$$

Then substitution of Eqs. (10) into the dynamic Eq. (7) gives:

$$\frac{\partial^2\Psi}{\partial(x^2)^2}+A\frac{\partial\Psi}{\partial x^2}=B \tag{11}$$

where

$$A=-\frac{\partial}{\partial x^2}\left[\ln\left(\sqrt{\frac{a_{22}}{a_{11}}}\,\tau\sin\theta_{12}\right)\right]$$

$$B=\frac{\partial\ln\rho}{\partial x^2}\frac{\partial\Psi}{\partial x^2}+\sqrt{\frac{a_{22}}{a_{11}}}\,\tau\sin\theta_{12}\rho(C+D+E)$$

with C, D, and E denoting the three terms on the right side of Eq. (9). By using either an equal or unequal spacing three-point differentiation formula and with the difference of the Ψ values at the lower and upper boundary walls known, the set of differential equations obtained at the grid points along the x^2 line can be solved easily by a standard backward elimination technique.

Flowfield Matrix Method

Substituting Eqs. (10) into the dynamic Eq. (7) results in the following form of principal equation:

$$\frac{1}{a_{11}}\frac{\partial^2\Psi}{\partial(x^1)^2}-2\frac{\cos\theta_{12}}{\sqrt{a_{11}a_{22}}}\frac{\partial^2\Psi}{\partial x^1\partial x^2}+\frac{1}{a_{22}}\frac{\partial^2\Psi}{\partial(x^2)^2}$$
$$+\frac{J}{\sqrt{a_{11}}}\frac{\partial\Psi}{\partial x^1}+\frac{K}{\sqrt{a_{22}}}\frac{\partial\Psi}{\partial x^2}=\frac{\tau\sin\theta_{12}}{\sqrt{a_{11}a_{22}}}\rho C \tag{12}$$

where

$$J=-\frac{\partial\ln(\sqrt{a_{11}/a_{22}}\,\tau\sin\theta_{12})}{\sqrt{a_{11}}\partial x^1}+\frac{\cos\theta_{12}}{\sqrt{a_{22}}}\frac{\partial\ln\tau}{\partial x^2}$$
$$+\frac{1}{\sin\theta_{12}\sqrt{a_{22}}}\frac{\partial\theta_{12}}{\partial x^2}-\left(\frac{1}{\sqrt{a_{11}}}\frac{\partial\ln\rho}{\partial x^1}-\frac{\cos\theta_{12}}{\sqrt{a_{22}}}\frac{\partial\ln\rho}{\partial x^2}\right)$$

$$K=-\frac{\partial\ln(\sqrt{a_{22}/a_{11}}\,\tau\sin\theta_{12})}{\sqrt{a_{22}}\partial x^2}+\frac{\cos\theta_{12}}{\sqrt{a_{11}}}\frac{\partial\ln\tau}{\partial x^1}$$
$$+\frac{1}{\sin\theta_{12}\sqrt{a_{11}}}\frac{\partial\theta_{12}}{\partial x^1}-\left(\frac{1}{\sqrt{a_{22}}}\frac{\partial\ln\rho}{\partial x^2}-\frac{\cos\theta_{12}}{\sqrt{a_{11}}}\frac{\partial\ln\rho}{\partial x^1}\right)$$

Equation (12) is a second-order quasilinear partial differential equation in Ψ. In the design problem, since the variation of $(V_\theta r)$ is prescribed by the designer, it is considered as a known value in the equation. The nonlinear terms include the variable ρ which is treated in the following manner: $\Psi^{(\nu)}$ obtained in the νth iteration and $\rho^{(\nu-1)}$ obtained in the $(\nu-1)$th iteration are used to compute the relative velocity W and the temperature T. Then $\rho^{(\nu)}$ is used for the solution of $\Psi^{(\nu+1)}$. The accuracy of ρ is increased with the accuracy of Ψ during successive approximation.

Equation (12) is an elliptic partial differential equation when the meridional velocity W_l is less than the velocity of sound and is solved as a boundary value problem of the mixed type.

A number of methods can be used to solve this set of equations for Ψ—e.g., point relaxation, improved point relaxation, line relaxation, and direct solution. The line relaxation technique will be discussed first in connection with S_2 surface, and the direct solution will be discussed in the next section in connecton with the S_1 surface. This line relaxation technique is also called the block iteration method.

Theoretically speaking, the condition of integrability provides an extra condition for the determination of the f_2 term on the right-hand side of Eq. (12).[1,4] f_2 is the covariant component of the force exerted on the stream surface by the surrounding gas F. In an actual design problem, one may also calculate the f_2 term in the νth iteration from the relevant quantities obtained in the $(\nu-1)$th iteration as follows:

$$f_2^{(\nu)}=-rF_\varphi\frac{\partial\varphi^{(\nu-1)}}{\partial x^2}$$
$$=-\left[\frac{W^1\partial(V_\theta r)}{\sqrt{a_{11}}\partial x^1}-\frac{W^2\partial(V_\theta r)}{\sqrt{a_{22}}\partial x^2}\right]^{(\nu-1)}\frac{\partial\varphi^{(\nu-1)}}{\partial x^2} \tag{13}$$

The typical effect of the blade tilt angle μ of the stator blade on the radial distribution of meridional velocity V_l is shown in Fig. 4. The tilt angle is the angle between the stacking line of the blade section and radial line at the same j station. It is seen that when the tilt angle is not equal to zero, the effect of the corresponding term f_2 on the flow distribution is obvious.

Only when the stacking line is in the radial direction are the term f_2 and its effects small. In Fig. 5 are shown the effects of blade thickness on the meridional velocity V_l and flow angle β along a station at 40% of the chord in the passage of the first stage rotor.

Computer Program for Relative Stream Surface S_1

In the design of turbomachine blades on the basis of three-dimensional flow, the usual procedure of geometrically constructing the contour of a number of blade sections according to the inlet and outlet angles obtained in the S_2 solution is not satisfactory, because the blade contour thus obtained does not ensure that the flow condition calculated on the S_2 surface will be realized. Furthermore, the geometrical blade contour is usually restricted to a number of circular arcs laid out on conical surfaces. Usually they do not give rise to desirable velocity and pressure distributions over the blade surface. In order to improve the design, the S_1 computer program is used to analyze the flow on the actual S_1 flow surface to find the distributions of velocity and pressure over the blade surface, so that from an analysis of these distributions modification of the blade contour can be made. The output of the program also gives flow data along the

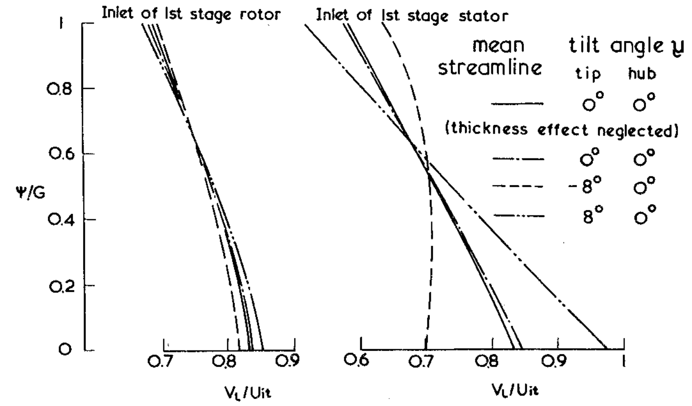

Fig. 4 Effect of f_2 term on V_l distribution in radial direction.

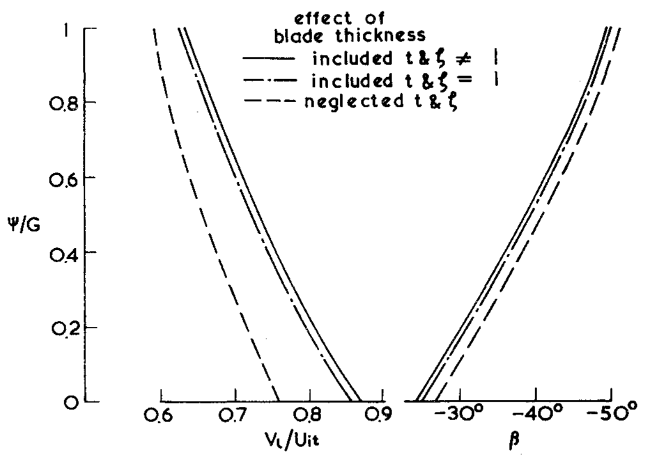

Fig. 5 Effects of blade thickness on V_l and β at the station in passage of rotor.

streamlines which form the S_2 surfaces, and also the distance between the neighboring S_2 surfaces. If these results do not agree with those obtained in preceding solution of the S_2 surfaces, they should be used for a corrected solution of the S_2 surface flow. Two S_1 programs for subsonic inlet flow and one program for supersonic inlet will be described in the following sections.

Matrix Program for Subsonic Flow

For axial flow turbomachines, the axial distance across a blade row is relatively small, so that the assumption of the S_1 surface as a surface of revolution is usually acceptable. The principal equation expressed in terms of the stream function in the present coordinate system is‡

$$\frac{1}{a_{11}}\frac{\partial^2\Psi}{\partial(x^1)^2}-2\frac{\cos\theta_{12}}{\sqrt{a_{11}a_{22}}}\frac{\partial^2\Psi}{\partial x^1\partial x^2}+\frac{1}{a_{22}}\frac{\partial^2\Psi}{\partial(x^2)^2}+\frac{J}{\sqrt{a_{11}}}\frac{\partial\Psi}{\partial x^1}+\frac{K}{\sqrt{a_{22}}}\frac{\partial\Psi}{\partial x^2}=M \tag{14}$$

where

$$J=-\frac{\partial\ln(\sqrt{a_{11}/a_{22}}\,\tau\sin\theta_{12})}{\sqrt{a_{11}}\partial x^1}+\frac{\cos\theta_{12}\partial\ln\tau}{\sqrt{a_{22}}\partial x^2}+\frac{1}{\sin\theta_{12}\sqrt{a_{22}}}\frac{\partial\theta_{12}}{\partial x^2}$$

$$K=-\frac{\partial\ln(\sqrt{a_{22}/a_{11}}\,\tau\sin\theta_{12})}{\sqrt{a_{22}}\partial x^2}+\frac{\cos\theta_{12}\partial\ln\tau}{\sqrt{a_{11}}\partial x^1}+\frac{1}{\sin\theta_{12}\sqrt{a_{11}}}\frac{\partial\theta_{12}}{\partial x^1}$$

$$M=\left(\frac{1}{\sqrt{a_{11}}}\frac{\partial\ln\rho}{\partial x^1}-\frac{\cos\theta_{12}}{\sqrt{a_{22}}}\frac{\partial\ln\rho}{\partial x^2}\right)\frac{1}{\sqrt{a_{11}}}\frac{\partial\Psi}{\partial x^1}+\left(\frac{1}{\sqrt{a_{22}}}\frac{\partial\ln\rho}{\partial x^2}-\frac{\cos\theta_{12}}{\sqrt{a_{11}}}\frac{\partial\ln\rho}{\partial x^1}\right)\frac{1}{\sqrt{a_{22}}}\frac{\partial\Psi}{\partial x^2}-2\omega\tau\rho\sin\sigma\sin^2\theta_{12}$$

Equation (14) is a second-order quasilinear partial differential equation. In the succesive approximate solution of this equation, the terms on the right hand side for the νth solution are evaluated from the $(\nu-1)$th solution and are therefore considered as known values in the νth solution. Thus, the criterion for the type of the equation becomes

$$\frac{\cos^2\theta_{12}}{a_{11}a_{22}}-\frac{1}{a_{11}}\frac{1}{a_{22}}<0 \tag{15}$$

which means that the equation is always elliptic. This result is due to the present treatment of the density term in the iterative solution. Actually, the density term is related to the second-order partial derivatives of Ψ; therefore, the criterion should include these derivatives, and the equation will be elliptic or hyperbolic when the relative velocity is subsonic or supersonic as discussed in Ref. 1. It is known from aerothermodynamic relations that, for a given distribution of Ψ, there are two solutions of density according to whether the flow is subsonic or supersonic. For the subsonic S_1 program, only the density value corresponding to the subsonic velocity is used. This is consistent with the elliptic criterion mentioned above. An automatic alarm is provided in the program wherever supersonic local velocity appears.

The direct matrix solution described in Ref. 1 is used in the present program.

It should be noted that the treatment of the density term here is different than that in the S_2 program. Here, all the density terms are kept on the right side of the equation so that all elements in the matrix M are only geometrical functions of the coordinates and the given distance τ_1 between the neighboring S_1 surfaces. Thus, the triangular decomposition has to be done only once for one geometrical configuration. This feature is especially desirable when solutions for different inlet conditions are desired for a given blade section. During each calculation, the terms on the right side are successively corrected. Our experience is that it usually takes only about 10 iterations to obtain a converging solution of acceptable engineering accuracy.

As is well known, the disadvantage of the matrix direct solution is the requirement for a large internal storage capacity of the matrices. We have greatly reduced this storage requirement by the following two devices:

1) The nonzero elements in the matrix M are situated around the diagonal line (Fig. 6). We assign j' numbers from left to right for the elements running parallel to the diagonal line, and assign k' numbers along the downward direction of the diagonal line. For the grid pattern shown in Fig. 3 and with the use of three-point central differential formulas, the length and width of the band of nonzero elements is respectively mn and $4n-1$, and the storage is $N=(4n-1)mn$ (Fig. 6a).

2) Far upstream and far downstream of the blades, the existence of the periodic boundary condition makes the width of the band twice longer. In order to minimize the storage requirement, an extra row of grid points are added, as shown in Fig. 3. These points are not calculation points but are added to the grid as fictitious points for the purpose of reducing the width of the band (Fig. 6b). Then, the width of the band becomes $2n+7$. The number of elements stored is now $2n^2m+(11n+14)m$ and the amount reduced is $2n^2m-(12n+14)m$. In this way, the internal storage is reduced nearly to two-thirds. The advantage increases with n.

‡This equation was first obtained by the senior author in 1970.

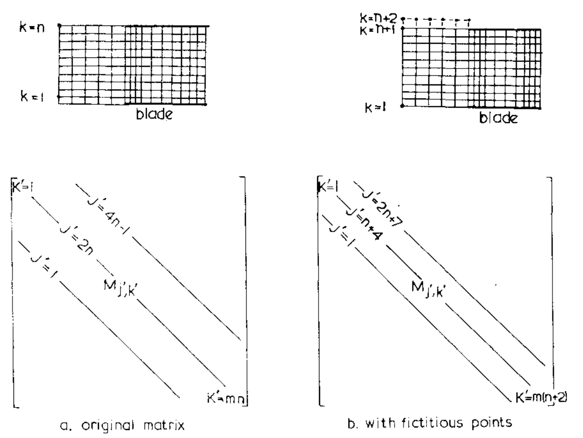

Fig. 6 Nonzero elements in coefficient matrix.

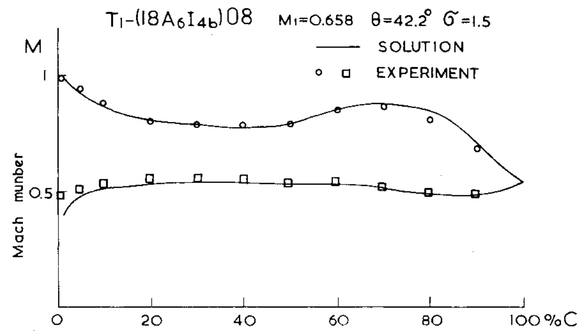

Fig. 7 Blade surface Mach number distribution.

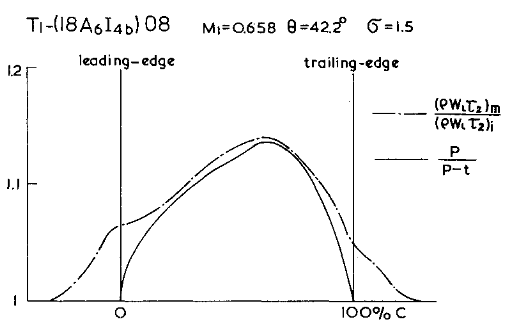

Fig. 8 Effect of blade thickness on $(\rho W_l \tau_2)$.

This program can be used to analyze the flow past a cascade of a blade section on an arbitrary stream surface of revolution. In Figs. 7 and 8 are shown the results of calculations of flow past a cascade of compressor blades composed of $T_l-(18A_6I_{4b})08$ sections. The comparison of the calculated solutions with the experimental values[11] in Fig. 7 is quite good. It is seen that due to the relatively rearward location of the maximum thickness, the velocity peak near the leading edge is reduced, and there is a second peak downstream.

The variations of $(\rho W_l \tau_2)_m/(\rho W_l \tau_2)_i$ along the central streamline is shown in Fig. 8. It is interesting to note that they are similar in nature to those obtained in turbine cascades published in Refs. 2, 6, and 8. They are to be used as input values for S_2 calculation.

Mean Streamline Method Program

The mean streamline method outlined in Refs. 6 and 8 has been programmed. Recently, this method has been to possess the ability to compute high subsonic inlet flows with local supersonic Mach numbers. The close agreement with experimental values obtained in Ref. 12 is very encouraging. We

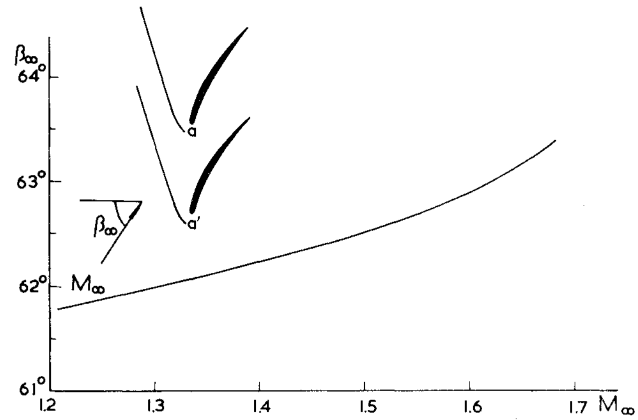

Fig. 9 Variation of unique incidence with M_∞ for cascade of multicircular arc blade.

are now using the program either to design the blade section alone a S_l surface of revolution based upon the flow variation along the central streamline obtained in the central S_2 solution, or to modify the blade contour obtained by the usual geometrical construction on a developed conical surface.

Method of Characteristics Program for Supersonic Inlet Flow

We have also programmed the method of characteristics solution for supersonic flow given in Ref. 13. The effects of the variations in the distance between two neighboring S_l surfaces and the radius are taken into account. For simplicity, the shape of the bow wave is taken to be that given in Ref. 14, since it has been shown to be true for supersonic flow of the Mach number range encountered in transonic compressors. In the program, the coefficients used in the calculation are the average values between the two connecting points obtained after iteration. Interpolation is made on a two-dimensional basis. The calculation is started with assumed values of Mach number and angle of inlet flow, and extended along the circumferential direction several pitches until the solution of one pitch length repeats itself. Then the uniform condition far upstream is determined by the use of the following set of equations of continuity, momentum, and energy:

Continuity:

$$\frac{2\pi r_\infty \tau_\infty}{N}\left(\rho W \cos\beta\right)_\infty = \int_{\varphi_a}^{\varphi_{a'}} (\rho W \cos\beta \tau r)\,d\varphi \qquad (16)$$

Momentum, axial:

$$\frac{2\pi r_\infty \tau_\infty}{N}\left(p+\frac{\rho W^2}{g}\cos^2\beta\right)_\infty = \int_{\varphi_a}^{\varphi_{a'}} \left[\left(p+\frac{\rho W^2}{g}\cos^2\beta\right)\tau r\right]d\varphi \qquad (17)$$

Momentum, circumferential:

$$\frac{2\pi r_\infty^2 \tau_\infty}{N}\left(\rho W^2 \sin\beta\cos\beta\right)_\infty = \int_{\varphi_a}^{\varphi_{a'}} (r^2\tau\rho W^2 \cos\beta\sin\beta)\,d\varphi \qquad (18)$$

Energy:

$$h_\infty + \frac{1}{2gJ}(W^2-\omega^2 r^2)_\infty = h_a + \frac{1}{2gJ}(W_a^2-\omega^2 r_a^2) \qquad (19)$$

State:

$$p = R\rho T$$

The value of β_∞ obtained is then the one corresponding to the unique incidence for the incoming M_∞. Figure 9 shows a typical variation of β_∞ with M_∞ obtained in this manner for a cascade of a multicircular arc blade.

Comparison between Different Methods of Solution

We have purposely employed different methods of solution in the S_1 and S_2 programs described above in order to compare the effectiveness of these methods in the solution of turbomachine problems. These methods are usually divided into the following two groups:

Streamline Extension Method

The advantages of this method are: computation is relatively simple, and the requirement for internal storage is small. However, due to the large number of terms on the right side of the equation and the necessity to make successive corrections of the position and shape of the projection of the streamline on the meridional plane and the physical quantities on the streamline, the speed on convergence of the solution is poor in most cases. It is then necessary to employ a very small relaxation factor α on the successive corrections of the streamline position to avoid divergence. From relevant analysis and our actual calculations, the grid spacing Δm in the direction of the streamline is the major factor affecting convergence. Roughly, the relaxation factor α is related to Δm as follows:

$$\alpha \propto \Delta m^2$$

Our experience with a two-stage compressor shows that:

1) When calculation stations are located only in the gap between blade rows, α can be as large as 0.3 and after 13 iterations the error in Ψ (varies between 0 and 1) is less than 10^{-3}.

2) When there are three more stations added in the blade passage, the maximum value of α is 0.04, and 80 iterations are required to attain the same accuracy in Ψ.

3) When the stations within the blade passage is increased to 6, the solution diverges when α is greater than 0.01, and 273 iterations are required to attain an accuracy $0.58 \cdot 10^{-2}$ with α equal to 0.01.

From this and other similar calculations, we feel that this method of solution is more suitable for performance calculation, in which calculating stations are required only in the gap between the blade rows.

The mean streamline method employed in the S_1 surface seems to work quite well with compressor cascade of moderate solidity and low local supersonic Mach number. Of course, it has the same disadvantage as other streamline extension methods, in that it will not give detailed flow variations around the leading and trailing edges of the blade. Our experience indicates that the extension of solution in a single step by the use of three terms in the series without using the intermediate streamline is adequate.

Flowfield Matrix Method

The advantage of the flowfield matrix method is that when the calculating stations are placed in the blade passage, the solution converges very well. When the line relaxation technique is used, the over-relaxation factor can always be employed. When the height/width ratio of the grid spacing is equal to 5, the factor α obtainable with the streamline extension technique is on the order of 10^{-2}, whereas in the line relaxation technique, the over-relaxation factor is in the range of 1-1.65, and only 15-20 iterations are required to reach the accuracy of ordinary engineering calculations. Therefore, this is the technique finally adopted in our S_2 computer programs.

From the preceding discussion, it may be concluded that:

1)It is always desirable to reduce the number of terms on the right side of the principal equation, which are to be successively corrected, and to use values of the present cycle as much as possible. This is most important for quick convergence.

2) When calculating stations are placed within the blade passage, the flowfield matrix method converges faster than the streamline extension method. Also, the matrix direct solution is preferred to the line relaxation solution whenever the internal storage is big enough.

Successive Calculations of S_1 and S_2 for Three-Dimensional Blade Design

The S_1 and S_2 programs described above have been used for the design of an axial flow turbomachine blade based upon three-dimensional flow. As mentioned before, the assumption of the S_1 surface being one of revolution is considered rather accurate and thus acceptable. With this simplification, only one S_2 surface is needed to determine the shape of the S_1 surfaces. And it is natural to use the S_{2m} surface located somewhere in the midpart of the flow passage.

The design calculation starts with S_{2m}, according to the design mass flow, pressure ratio, hub and tip contours, which may be determined by the use of the simplified radial equilibrium program and optimized by the use of the Latin square technique. The whole procedure involved is indicated in Fig. 10. The content of the various operations are as follows:

1) Preparation for the S_{2m} calculations: the hub and tip contours in the meridional plane, the variations of $V_\theta r$, s, etc.

2) Solution of flow on the S_{2m} surface.

3) Examination of the aerodynamic limiting factors, such as inlet Mach number, diffusion factor, velocity ratio, etc.

4) If these are not satisfactory, step 2) is repeated with modified input values.

5) If the blade contour is determined by geometrical construction, the S_1 surface of revolution is approximated by the conical surface and curves such as circular arcs are laid out on the developed conical surface.

6) If the blade contour is determined by an aerodynamic method such as the mean streamline method, calculation can be made on the obtained surface of revolution.

7) The S_1 program described above is used to analyze the velocity and pressure distributions on the blade surface and the shape of the flow variation on the central streamline. If the velocity and pressure distribution are not satisfactory, some input data to the S_{2m} may be modified and the whole process repeated.

8) If the velocity and pressure distribution on the blade surface are satisfactory, but the shape of the flow distribution on the mean streamline is different from that used in step 2), S_{2m} should be repeated with these new values.

Our experience indicates that the successive iterative calculations of S_{2m} and S_1 converge rather satisfactorily. Usually, only three or four cycles of iteration were sufficient. Some of the results obtained in a typical three-dimensional

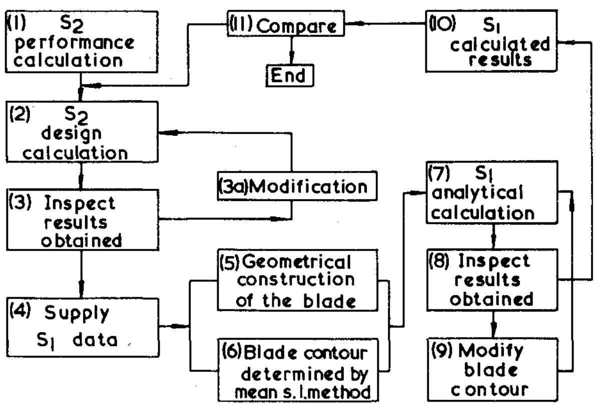

Fig. 10 Iterative procedure between S_1 and S_2.

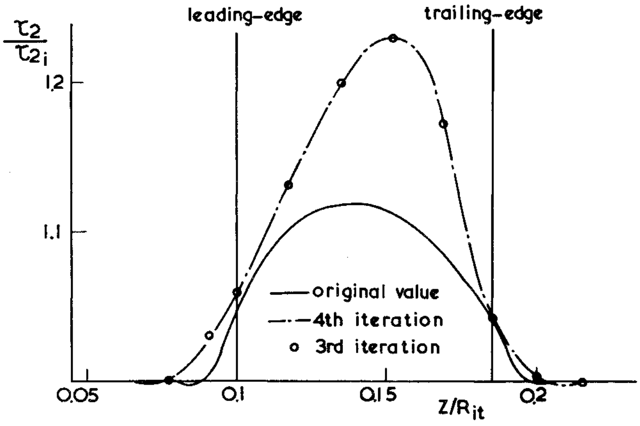

Fig. 11 Convergency of τ_2 (hub section).

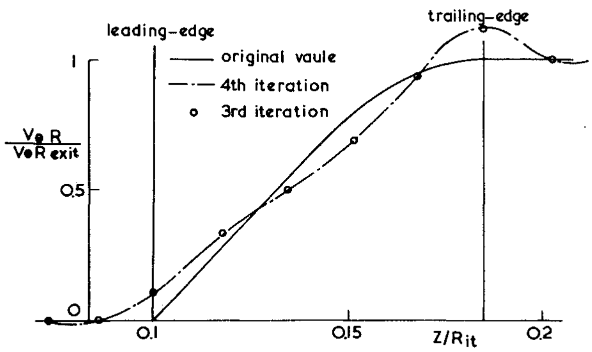

Fig. 12 Convergence of ($V_\theta r$) (hub section).

design of a single compressor rotor are shown in Figs. 11 and 12. They include the variations in thickness of S_2 filaments τ_2 and $V_\theta r$ along the central streamline. The convergence is seen to be satisfactory, and solution is usually obtained without difficulty. The design system described above for axial flow compressor blades is considered to be practical.

Conclusions

From our work on S_1 and S_2 computer programs, their application to design of axial flow turbomachine blades, and comparison with available experimental data, the following conclusions may be made:

1) The convergency of the iterative successive calculations of the S_1 and S_2 surfaces is satisfactory, and the blade design based upon three-dimensional flow is practical.

2) The employment of general nonorthogonal curvilinear coordinates and corresponding nonorthogonal velocity components helps to satisfy the arbitrary boundary conditions of the hub and tip walls and blade surfaces more satisfactorily, and converges more rapidly and accurately than the old orthogonal ones.

3) In the successive calculations of the S_1 and S_2 surfaces, the central (or mean) streamlines are the lines common to the two kinds of surfaces (the flow on these lines is representative of that in the whole channel, but is not equal to the circumferentially averaged value). Particular attention should be paid to the flow on these streamlines.

4) Among the different numerical methods we have used, we found that the principal equation expressed in terms of the stream function and solved by either the matrix direct method or the line relaxation technique is most satisfactory from the point of view of fast convergency and accuracy.

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